Modeling, Simulation, and Learning for Urban Crime Dynamics

Published in *Mathematical Models and Methods in Applied Sciences* (2026)

Baoli Hao[1], Kamrun Mily[2], Annalisa Quaini[2], Ming Zhong[2]

[1]Department of Applied Mathematics, Illinois Institute of Technology
[2]Department of Mathematics, University of Houston

Motivation

Why Mathematical Crime Modeling?

- Crime distributions exhibit low regularity and complex patterns
- Different crime types show distinct patterns of emergence, diffusion, and dissipation
- Goal: Develop quantitative tools to assist municipalities in directing community investments: from mechanistic modeling to learning parameters/dynamics from crime data
- We study residential burglary, where mobile offenders target stationary sites, with the additional ingredient of law enforcement deployment

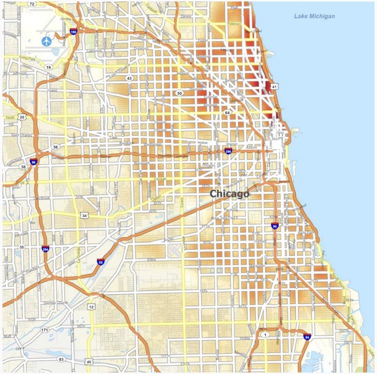
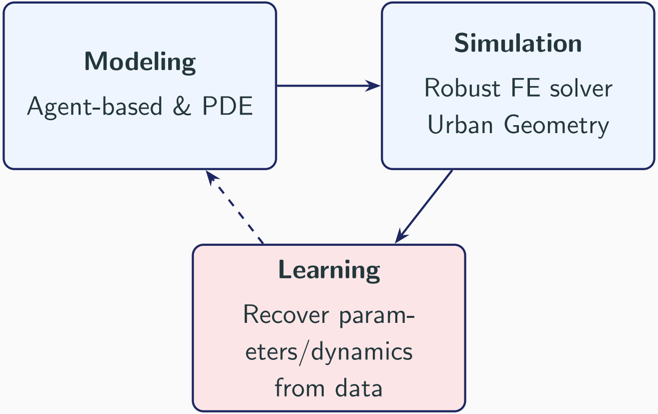


Figure 1: Chicago

Outline


Modeling
Agent-based & PDE
Simulation
Robust FE solver
Urban Geometry
Learning
Recover param-
eters/dynamics
from data

Modeling & Simulation

Urban Crime: Agent-based Model[1]

Setup:

- Domain $\Omega \subset \mathbb{R}^2$, Cartesian lattice with spacing h, time step Δt
- Attractiveness: $A_s(t) = A_s^{st} + B_s(t)$
- Crime probability (Poisson): $p_s(t) = 1 - e^{-A_s(t)\Delta t}$

Dynamic attractiveness:

$$B_s(t+\Delta t) = \left[(1-\eta)B_s + \frac{\eta}{4}\sum_{s'\sim s} B_{s'}\right](1-\omega\Delta t) + \theta E_s$$

Criminal population:

$$n_s(t+\Delta t) = A_s \sum_{s'\sim s} \frac{n_{s'}[1-p_{s'}]}{T_{s'}} + \Gamma\Delta t$$

Key parameters: η (neighborhood effects), ω (decay), θ (victimization boost), Γ (criminal generation), $E_s = n_s p_s$ (expected burglaries at site s)

[1] Short et al., *Mathematical Models and Methods in Applied Sciences*, 2008.

From Agent-based Model to PDE Model (Continuum Limit)

Taking $h, \Delta t \to 0$ with $\frac{h^2}{\Delta t} = D$ and $\theta \Delta t = \epsilon$ fixed, we obtain the **non-dimensional system:**

$$\frac{\partial A}{\partial t} - \eta \Delta A + A - \rho A = -\eta \Delta A^{st} + A^{st},$$

$$\frac{\partial \rho}{\partial t} - \nabla \cdot \left(\nabla \rho - \frac{2 \nabla A}{A} \rho \right) + \rho A = \frac{\Gamma \theta}{\omega^2} \; =: \; \rho^{src},$$

with **natural boundary conditions** (Neumann): $\nabla A \cdot \mathbf{n} = 0, \;\; \nabla \rho \cdot \mathbf{n} = 0$ on $\partial \Omega$, and initial conditions with optional noise:

$$A(\boldsymbol{x}, 0) = A^{st}(\boldsymbol{x}) + B_0(\boldsymbol{x}) + \chi_\delta(\boldsymbol{x}) \xi_B(\boldsymbol{x}), \quad \rho(\boldsymbol{x}, 0) = \rho_0(\boldsymbol{x}) + \chi_\delta(\boldsymbol{x}) \xi_\rho(\boldsymbol{x}).$$

- Similar to **Keller–Segel chemotaxis**.
- Key unknowns: η (neighborhood effects) and ρ^{src} (criminal generation).

Finite Element Solver

Why Finite Elements:

- Treats Neumann boundary conditions in the variational formulation naturally.
- Enables realistic geometries flexibly.
- Captures spatial heterogeneity accurately.

Our approach: Iterative partitioned scheme

1. Solve for A^{k+1} (linear)
2. Solve for ρ^{k+1} using A^{k+1} (linear)
3. Convergence check; iterate if needed

With police:

$+$Update H^{k+1} (algebraic) $+$ Solve π^{k+1} (linear)

Challenge:

- Strongly nonlinear coupling ($2\nabla A/A \cdot \rho$)
- Fully coupled Newton: slow convergence, dense Jacobian at scale

Why it works well:

- Linear substep $\rightarrow$ sparse direct solve
- Strongly-coupled: iterates until convergence, unlike loosely-coupled schemes that can diverge
- **1–6 avg. iterations** across 3 orders of magnitude in tolerance and mesh size

Open-source code on **Github**: `https://github.com/baolihao/UrbanCrime-Sim`

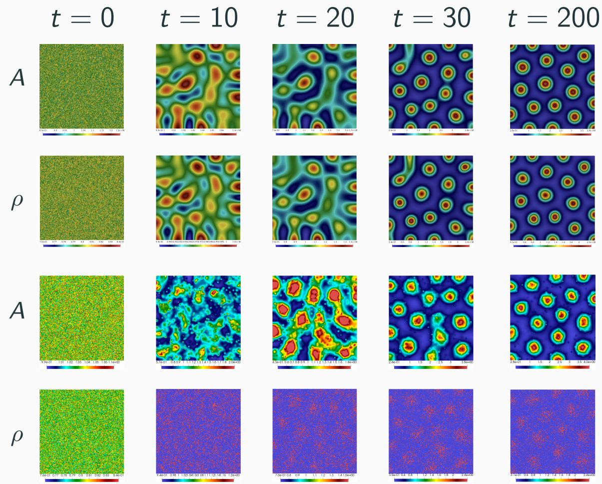


Figure 2: Case($\eta = 0.3$). Top: PDE model. Bottom: Agent-based model. Hotspot sizes and values match.

Without Deterrence/Control: Chicago with Highway I-90

- Realistic city boundary
- Highway modeled via spatially heterogeneous $A^{st}(\boldsymbol{x})$ and $\Gamma(\boldsymbol{x})$
- Persistent hotspots along the highway
- Smaller transient hotspots in parallel directions

Empirical support: highway openings $\rightarrow$ $\sim$5% rise in property crime [4]

Figure 3: Simulation: Chicago with Highway I-90

[4] Calamunci & Lonsky, *IZA Inst. Labor Econ. Rep.*, 2022.

Extended Model [7]

Third agent type: Control agents ($m_s(t)$ at site s, total $\sum_s m_s = M$ conserved)

Effect on crime:

- Control agents reduce crime probability:

$$p_s^c(t) = 1 - e^{-A_s e^{-\beta m_s / h^2} \Delta t}$$

- Σ: arrest probability per burglary
- Criminal regeneration dampened by $e^{-\beta m_s / h^2}$

Control agents movement:

- Biased random walk toward high-crime areas
- Based on time-lagged signal:

$$H_s(t+\Delta t) = \left(1 - \frac{\Delta t}{\tau}\right) H_s + \frac{\Delta t}{\tau} S_s$$

- Delay τ: time for data to reach decision-makers; Expected number of crimes per time per space: $S_s(t) = \frac{E_s(t)}{h^2 \Delta t}$

[7] Hao et al., *arxiv: https://arxiv.org/pdf/2605.17709*, 2026.

PDE Model

4 coupled equations (non-dimensional):

$$\frac{\partial A}{\partial t} - \eta \Delta A + A - \rho A e^{-\pi} = -\eta \Delta A^{st} + A^{st},$$
$$\frac{\partial \rho}{\partial t} - \nabla \cdot \left(\nabla \rho - \frac{2 \nabla A}{A} \rho \right) + \rho A e^{-\pi} = \rho^{src}(1 - \Sigma) e^{-\pi},$$
$$\frac{\partial \pi}{\partial t} - \nabla \cdot \left(\nabla \pi - \frac{2 \nabla H}{H} \pi \right) = 0,$$
$$\frac{\partial H}{\partial t} = \frac{1}{\tau} \left(\rho A e^{-\pi} - H \right).$$

Key features:

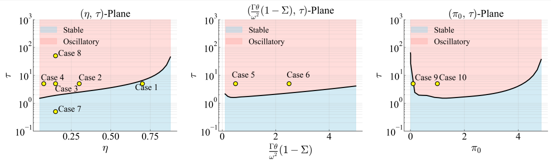
- π: police density; H: lagged crime signal
- $e^{-\pi}$ suppresses crime & regeneration
- No source/sink in the control eq. $\rightarrow$ conservation

Parameter reduction:

- 3 free parameters: η, $\rho^{src}(1-\Sigma)$, τ
- Short et al. 2010 (Fixed Police), Zipkin et al. 2014 (Optimal Strategy) are special cases (π prescribed)

Delay-Induced Oscillatory Instability

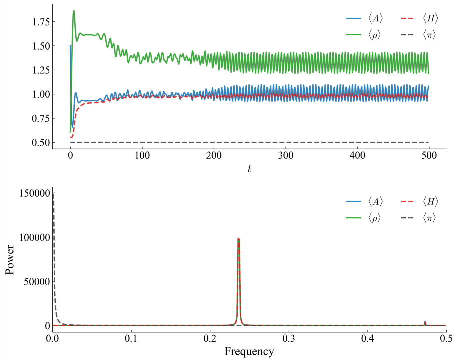
- Perturb homogeneous equilibrium in Neumann mode $\Rightarrow$ 4×4 eigenproblem
- **Routh–Hurwitz** on the quartic: as τ grows, a complex pair $\lambda = \pm i\omega_0$ crosses into $\mathrm{Re}\,\lambda > 0 \Rightarrow$ Hopf, critical delay τ_c^*
- Panels: τ vs η, $\Gamma\theta(1-\Sigma)/\omega^2$, π_0. stable below τ_c^*, oscillatory above



Larger η raises τ_c^*. $\tau_c^* \to \infty$ as $\pi_0 \to 0$ (no police, no oscillation)

Numerical confirmation

- $\langle A\rangle, \langle\rho\rangle, \langle H\rangle$: persistent bounded oscillations. $\langle\pi\rangle$ constant (police conserved)
- Dominant period ≈ 4.24



No patrol $\to$ steady hotspots **With patrol** $\to$ oscillating hotspots

With Deterrence/Control: Hotspot Dynamics Over One Cycle

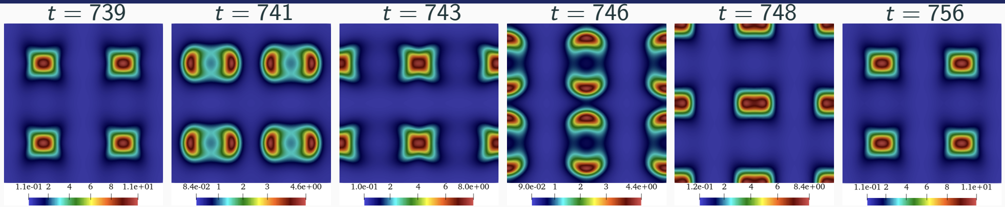


Figure 4: Expected crime rate $S = \rho A e^{-\pi}$ over $\sim$4 periods ($t = 739$ to 756).

- Hotspots cyclically split, migrate, and reform
- Phase portrait $(\langle A \rangle, \langle \rho \rangle)$: closed loop. A and ρ oscillate out of phase due to lagged crime–control feedback
- Phase portrait $(\langle H \rangle, \langle S \rangle)$: Delayed response
- τ acts as a finite-memory effect: key bifurcation parameter

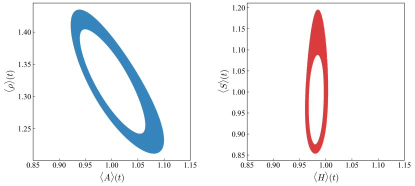
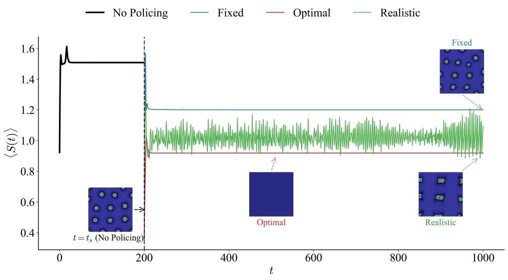


Figure 5: Phase portraits

Comparison of policing strategies



Deploy at t_s, $M = 50$ (total police), $\eta = 0.15$.

Information availability shapes the **level** *and* the **space–time structure** of crime.

- **Optimal** [5]: uniform hotspots, lowest crime, but needs real-time global optimization: re-solve every step, $\pi = \arg\min \int_\Omega e^{-\pi} \rho A \, d\mathbf{x}, \ \int_\Omega \pi = M$.
- **Fixed** [6]: deploy once at t_s from local attractiveness, then frozen, resulting in displaced hotspots, no data needed, but cannot follow displaced hotspots.
- **Realistic** ($\tau = 5$): moving hotspots, continuous yet delayed response. Sustained oscillations.

[5] Zipkin, Short, Bertozzi. *Discrete Contin. Dyn. Syst. B*, 2014.
[6] Short, Bertozzi, Brantingham. *SIAM J. Appl. Dyn. Syst.*, 2010.

Learning

The Inverse Problem

Setting: Given (noisy) observation data from the PDE model/Real data, can we recover the model parameters/hidden dynamics?

Target parameters:

- η — strength of neighborhood effects (diffusion coefficient)
- $\rho^{src} = \Gamma\theta/\omega^2$ — criminal source term

Why this matters:

- For real-city applications, parameters must be estimated from data
- Sensitivity to η and ρ^{src} is high (controls hotspot formation)
- Two complementary approaches developed:

Method 1: Weak-form + RQMC	**Method 2: PINN**
mesh-free, linear solve	neural network + autograd

Method 1: Weak-Form Estimation

Idea: Multiply each PDE by test functions $\varphi_i(t, \boldsymbol{x})$, integrate over $[0, T] \times \Omega$, integration by parts transfers all derivatives onto φ.

1: Multiply each PDE by test functions: $\varphi_{n,m,l}(t, x, y) = \sin\left(\frac{n\pi t}{T}\right) \sin^2\left(\frac{m\pi x}{L_x}\right) \sin^2\left(\frac{l\pi y}{L_y}\right)$, which vanish on $\partial([0, T] \times \Omega)$ with zero normal derivative. Expand the unknown fields in a 2D tensor-product basis $\{\psi_j(\mathbf{x})\}_{j=1}^J$: $\quad \eta(\mathbf{x}) = \sum_{j=1}^J \alpha_j \, \psi_j(\mathbf{x})$, $\rho^{src}(\mathbf{x}) = \sum_{j=1}^J \beta_j \, \psi_j(\mathbf{x})$.

2: Substituting into the weak form and integrating by parts gives, for each test mode i:

$$\underbrace{\iint \psi_j \, \nabla A \cdot \nabla \varphi_i}_{M_{ij}^{(1)}} \cdot \alpha_j = \underbrace{\iint A \, \partial_t \varphi_i - \iint A(1-\rho)\varphi_i + A^{st} \iint \varphi_i}_{r_i^{(1)}}, \qquad \underbrace{\iint \psi_j \, \varphi_i}_{M_{ij}^{(2)}} \cdot \beta_j = \underbrace{-\iint \rho \, \partial_t \varphi_i + \iint \left(\nabla \rho - \tfrac{2\rho}{A} \nabla A\right) \cdot \nabla \varphi_i + \iint \rho A \, \varphi_i}_{r_i^{(2)}}.$$

3: Stacking $N_{\text{modes}} \gg J$ test functions yields two linear systems solved by least squares:

$$\alpha = \arg\min_{\alpha} \|M^{(1)}\alpha - r^{(1)}\|^2, \qquad \beta = \arg\min_{\beta} \|M^{(2)}\beta - r^{(2)}\|^2.$$

Remark: Integrals via Monte-Carlo method, enabling mesh-free evaluation.

Method 2: Physics-Informed Neural Network (PINN)

Architecture:

- Input: $(t, x, y) \to$ 6-layer MLP (128 hidden) $\to (A, \rho)$
- $\eta,\ \rho^{src}$: learnable

Loss:

$$\mathcal{L} = \underbrace{\sum_i \left[(A_i^{pred} - A_i^{obs})^2 + (\rho_i^{pred} - \rho_i^{obs})^2\right]}_{\mathcal{L}_{\text{data}}} + \underbrace{\sum_j w_j \left[r_1^2 + r_2^2\right]}_{\mathcal{L}_{\text{PDE}}}$$

- $r_1 = \frac{\partial A}{\partial t} - \eta \Delta A + A - \rho A + \eta \Delta A^{st} - A^{st}$,
  $r_2 = \frac{\partial \rho}{\partial t} - \nabla \cdot \left(\nabla \rho - \frac{2\nabla A}{A} \rho\right) + \rho A - \ \rho^{src}$
- Strong-form residuals r_1, r_2 (PDE equations) via autograd

Parameter Recovery: Results

	True value	Weak-form (RQMC)	PINN
η	0.15	0.1509 ± 0.00006	0.1497
ρ^{src}	1.50	1.5064 ± 0.00014	1.4983

Weak-form:

- η: relative error $\sim 0.1\%$, larger variance (diffusion harder to pin down)
- 16 RQMC scrambles, seconds to run

PINN:

- Both parameters within $\sim 0.1\%$ of true values
- No $\nabla\rho$ data needed
- $\sim$50k epochs, training is slow

Parameter Recovery: Results

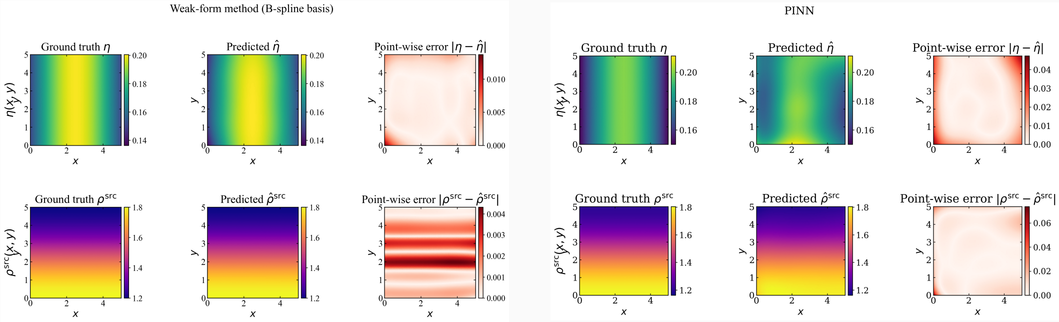


Figure 6: Results obtained using the weak-form method (left) and PINN (right). The ground-truth fields are given by $\eta = 0.15 + 0.05 \sin\left(\frac{\pi x}{L_x}\right)$ and $\rho^{\mathrm{src}} = 1.5 + 0.3 \cos\left(\frac{\pi y}{L_y}\right)$, with $L_x = L_y = 5$.

- 4×4 tensor-product B-spline basis are used in weak-form method.
- Boundary behavior is hard to captured.

Summary & Future Perspectives

Summary

Forward I[3]:

- FE framework for burglary PDE with Neumann BCs; robust solver; Chicago geometry

Forward II[7]:

- Dynamic police $\to$ oscillatory crime patterns; rich dynamics (τ as bifurcation parameter); rigorous analysis

Inverse (ongoing):

- **Weak-form + RQMC**: mesh-free, fast, built-in SE from 16 scrambles
- **PINN**: strong-form residuals via autograd, extendable to $\eta(\boldsymbol{x})$ naturally
- Both recover η and ρ^{src} from synthetic data

Open-source code: `github.com/baolihao/UrbanCrime-Sim`

[3] Hao et al., *Mathematical Models and Methods in Applied Sciences*, 2026.
[7] Hao et al., *arxiv: https://arxiv.org/pdf/2605.17709*, 2026.

Future Perspectives

Modeling & Simulation:

- More realistic geometries & real property data
- Kinetic description as mesoscopic bridge between agent-based and PDE models
- Rigorous analysis

Inverse / learning:

- Learn synthetic data with complex geometries and real crime data
- Extend to police model: learn τ
- Optimal adaptive modes and basis functions
- Learn dynamics from partial observations & physics

Thank You!

Questions?

Code: `github.com/baolihao/UrbanCrime-Sim`

Contact: `bhao2@hawk.illinoistech.edu`